Introductory notes on span

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Nov 25, 2010

Let X be a continuum (compact connected metric space) with metric d.

The notion of *span* of a continuum was introduced by A. Lelek in 1964, to study the following property, which holds for any chainable continuum X:

> $(*)$ If C is any continuum and f, g are continuous maps from C to X with $f(C) = g(C)$, then there is a point $t \in C$ with $f(t) = g(t)$.

Lelek considered the following quantity, designed to measure how far X is from satisfying the property $(*)$:

$$\sup_{C,f,g} \inf_{t \in C} d(f(t), g(t))$$

where the sup is taken over all continua C and all continuous maps $f, g : C \to X$ with $f(C) = g(C)$. Notice this number is 0 precisely when X satisfies the property $(*)$.

Problem 1. Show that the above number is equal to the following number, called the *span of* X:

$$\mathrm{Span}(X) = \sup_{Z} \inf_{(x,y) \in Z} d(x, y)$$

where the sup is taken over all subcontinua $Z \subset X \times X$ satisfying $\pi_1(Z) = \pi_2(Z)$ (here π_1 and π_2 are the first and second coordinate projections from $X \times X$ to X, respectively).

Problem 2. Observe that a continuum X has span equal to zero if and only if every subcontinuum Z of $X \times X$ satisfying $\pi_1(Z) = \pi_2(Z)$ meets the diagonal $\Delta X = \{(x, x) : x \in X\}$.

In the case that X is a graph, the span may be thought of as the largest number α such that two people can walk over the same portion of X while always staying at least distance α from one another.

Problem 3.

- Prove that the arc $[0, 1]$ has span equal to zero. *Hint: if Z is a subcontinuum of $[0,1] \times [0,1]$ with $\pi_1(Z) = \pi_2(Z) = X' \subseteq [0,1]$, then X' is itself an arc. Hence, we may as well assume that $\pi_1(Z) = \pi_2(Z) = [0,1]$.*

- Prove that the unit circle in the plane has span equal to 2 (its diameter).
- Prove that the *simple triod* $T = \{(x, 0) : -1 \leq x \leq 1\} \cup \{(0, y) : 0 \leq y \leq 1\}$ in the plane has span equal to 1.

Problem 4. Prove that if X is a chainable continuum, then X has span zero.

(Recall that a *chain cover* for X is a finite open cover $\langle U_1, \ldots, U_n\rangle$ ordered so that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$, and X is *chainable* if for any $\varepsilon > 0$ there is a chain cover for X of mesh $< \varepsilon$.)

In 1971, Lelek asked whether the converse of the above holds, i.e. whether all continua with span zero are chainable. I will discuss the construction of a counterexample for this question in the workshop.